Exercise 1

Prove the following:

$$\int_0^x \int_0^{x_1} (x-t)^3 u(x_1)\, dt\, dx_1 = \frac{1}{4}\int_0^x (x-t)^4 u(t)\, dt$$

[TYPO: The integrand should be $(x_1 - t)^3 u(t)$.]

Solution

From Right to Left

Let

$$G(x) = \frac{1}{4}\int_0^x (x-t)^4 u(t)\, dt.$$

Note that $G(0) = 0$. Differentiate both sides with respect to x and use the Leibnitz integration rule.

$$\begin{aligned} G'(x) &= \frac{1}{4}\frac{d}{dx}\int_0^x (x-t)^4 u(t)\, dt \\ &= \frac{1}{4}\int_0^x \frac{\partial}{\partial x}(x-t)^4 u(t)\, dt + \frac{1}{4}(0)^4 u(x)\cdot 1 - \frac{1}{4}x^4 u(0)\cdot 0 \\ &= \int_0^x (x-t)^3 u(t)\, dt \end{aligned}$$

Now integrate both sides with respect to x.

$$G(x) = \int^x \int_0^{x_1} (x_1-t)^3 u(t)\, dt\, dx_1 + C$$

Set the constant of integration and the lower limit of integration to 0 in order to satisfy $G(0) = 0$.

$$G(x) = \int_0^x \int_0^{x_1} (x_1-t)^3 u(t)\, dt\, dx_1$$

Therefore,

$$\int_0^x \int_0^{x_1} (x_1-t)^3 u(t)\, dt\, dx_1 = \frac{1}{4}\int_0^x (x-t)^4 u(t)\, dt.$$

From Left to Right

$$\int_0^x \int_0^{x_1} (x_1 - t)^3 u(t)\, dt\, dx_1$$

In order to evaluate this double integral, it's necessary to switch the order of integration because $u(t)$ is not given.

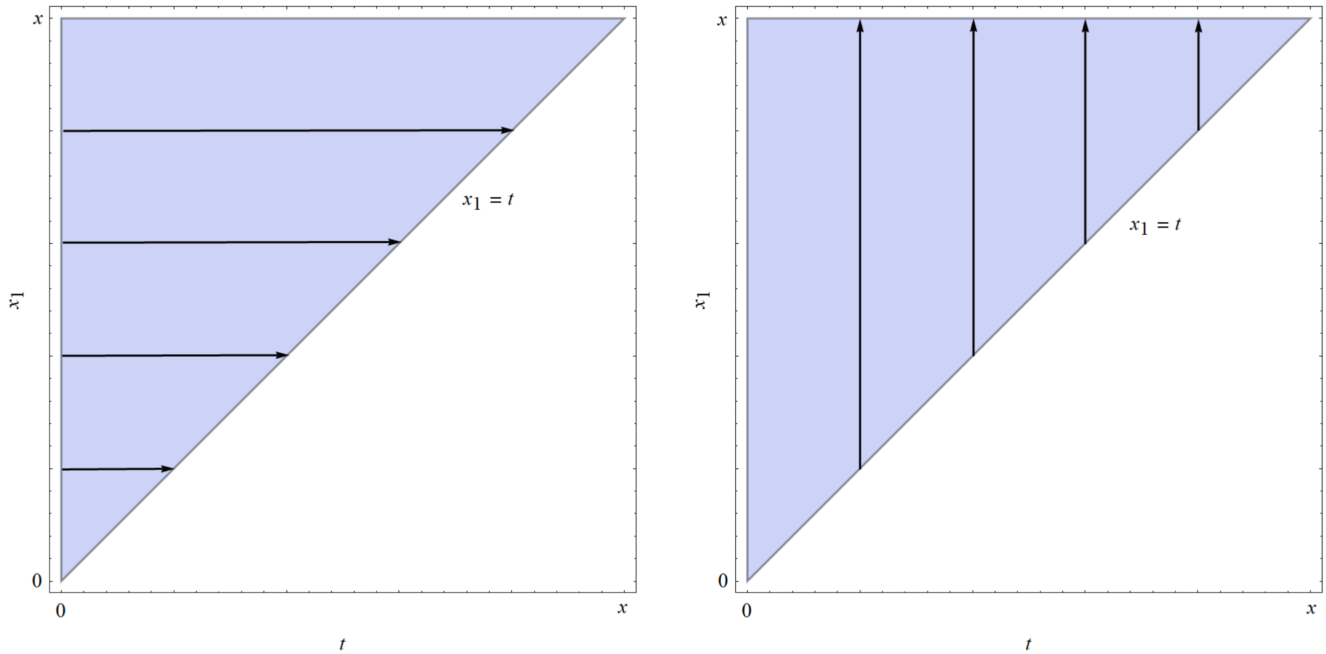

Figure 1: The current mode of integration in the tx_1-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$\begin{aligned}\int_0^x \int_0^{x_1} (x_1 - t)^3 u(t)\, dt\, dx_1 &= \int_0^x \int_t^x (x_1 - t)^3 u(t)\, dx_1\, dt \\ &= \int_0^x \left.\frac{(x_1 - t)^4}{4}\right|_t^x u(t)\, dt \\ &= \int_0^x \left[\frac{(x - t)^4}{4} - \frac{0^4}{4}\right] u(t)\, dt \\ &= \int_0^x \frac{(x - t)^4}{4} u(t)\, dt\end{aligned}$$

Therefore,

$$\int_0^x \int_0^{x_1} (x_1 - t)^3 u(t)\, dt\, dx_1 = \frac{1}{4}\int_0^x (x - t)^4 u(t)\, dt.$$